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2000 J. Phys. A: Math. Gen. 33 2569

(http://iopscience.iop.org/0305-4470/33/13/309)

J. Phys. A: Math. Gen. **33** (2000) 2569–2577. Printed in the UK PII: S0305-4470(00)03066-3

Involutive automorphisms and Iwasawa decomposition of affine Kac–Moody superalgebras

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Received 30 March 1999

Abstract. The involutive automorphisms of affine Kac–Moody superalgebras are computed from Satake superdiagrams corresponding to these algebras. These are then used to furnish a general treatment of the Iwasawa decomposition of these algebras. In particular, we consider $A^{(1)}(0,1)$ and $B^{(1)}(1,1)$ as representative examples for the purpose of illustration.

1. Introduction

The splendid success of the Kac–Moody [1, 2] algebras in providing a consistent and comprehensive framework for the development of formal aspects of a variety of theories serves as a motivating prelude to enhance the scope and range of applicability of these powerful mathematical techniques by way of exploiting the elegance of the concept of supersymmetry. It is, therefore, imperative to continue the scheme outlined in earlier investigations [3–5] for the study of affine Kac–Moody algebras to its supersymmetric version. This paper makes an attempt to study affine Kac–Moody superalgebras with special reference to the study of involutive root automorphisms and Iwasawa decompositions. The involutive automorphisms are obtained by constructing Satake superdiagrams for the affine Kac–Moody superalgebras. The procedure adopted here is essentially analogous to that of ordinary Lie algebras [5–7] and Lie superalgebras [4, 8–14] in general and affine Kac–Moody algebras [3, 15, 16] in particular, for which detailed and rather unambiguous prescriptions have already been outlined elsewhere. The main ingredients necessary for the evaluation of the involutive automorphisms constitute the root system and corresponding Dynkin diagrams from which to construct the Satake diagrams along with their supersymmetric versions. The root automorphisms for the affine Kac–Moody superalgebras are computed, in precisely the same manner as for the affine Kac–Moody and Lie superalgebras, with the help of the corresponding Satake superdiagrams for these superalgebras. These involutive automorphisms in turn provide a consistent framework to facilitate the Iwasawa decomposition [17] of Kac–Moody superalgebras. While the explicit individual calculations of the entire proliferation of these algebras constitute a daunting task, we shall restrict ourselves to the consideration of two representative examples, namely $A^{(1)}(0,1)$ and $B^{(1)}(1,1)$, to bring out the underlying notions in a unique manner. This choice of the untwisted versions is, however, dictated by reasons of mathematical simplicity.

The plan of the paper is as follows. Section 2 gives a brief outline of the essential steps for the construction of Satake superdiagrams from the Dynkin diagrams of affine untwisted Kac–Moody superalgebras and Iwasawa decomposition of these algebras. All possible Satake

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superdiagrams for $A^{(1)}(0, 1)$ and $B^{(1)}(1, 1)$ along with their root automorphisms have been computed and their Iwasawa decomposition carried out in detail in section 3. These are listed in tables 1 and 2, respectively. The structure of the inner automorphism required for the Iwasawa decomposition of affine Kac–Moody superalgebras is modified keeping in mind that both the roots $\alpha + n\delta$ and $-\alpha + n\delta$, $n \in \mathbb{Z}^+$ are positive roots in contrast to the fact that the positive root of the associated Lie superalgebra is only α. The modified inner automorphism and its properties along with the necessary mathematical formulation for calculating the elements of the nilpotent subalgebra $\underset{\sim}{n}$ have been discussed in detail in the appendix. Finally, a discussion and concluding remarks form the contents of section 4.

2. Satake superdiagrams and Iwasawa decomposition of affine Kac–Moody superalgebras

We begin this section by taking a look at the untwisted version of the affine Kac–Moody superalgebras and the construction of Satake superdiagrams.

The automorphisms of a simple Lie superalgebra $\mathcal{G}$ have been worked out in detail [3,8–14] and some of these results can be summarized as follows. Since any automorphism of a simple Lie superalgebra must respect grading, it will act on the reductive bosonic part $\mathcal{G}_0$ of the Lie superalgebra $\mathcal{G}$ as an automorphism of $\mathrm{Aut}(\mathcal{G}_0)$. Explicitly, an even (resp. odd) root is transformed by an automorphism into an even (resp. odd) root, otherwise grading will cease to be respected. The same argument can be extended to the affine Kac–Moody superalgebras, and construction of Satake superdiagrams from Dynkin diagrams for affine superalgebras is achieved with the help of the following prescriptions.

Let R be a root system of affine Kac–Moody superalgebra. For $\alpha \in R$, let $\bar{\alpha} = (-1)^{|\alpha|}\alpha - \sigma(\alpha)$, where $|\alpha|$ is the degree and $\sigma(\alpha)$ is the image of the root α under the automorphism σ. Let us introduce $R_- = \{\bar{\alpha} | \bar{\alpha} \neq 0, \alpha \in R\}$ and $R_0 = \{\alpha \in R | \bar{\alpha} = 0\}$. If B_- (resp. B) denotes the basis of R_- (resp. R) and B_0 a basis of R_0 then $B_0 = B \cap R_0$. If $B_- = B \backslash B_0 = \{\alpha_i\}$ and $B_0 = \{\beta_i\}$, then it can be shown that [3]

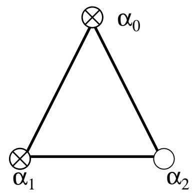
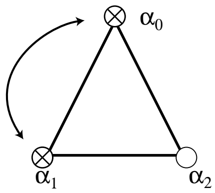
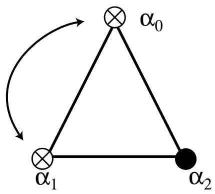
$$-\sigma(\alpha_i) = \alpha_{\pi(i)} + (-1)^{|\alpha_i|} \sum_l \eta_{il} \beta_l \tag{2.1}$$

where π is the involutive permutation of $\{0, 1, 2, \ldots, r\}$, and $(-1)^{|\alpha_i|}\eta_{il}$ are non-negative integers. We should note that $\sigma(\beta_l) = (-1)^{|\beta_l|}\beta_l$ and $\alpha + \sigma(\alpha) \notin R \ \forall \alpha \in R$. We can now associate B with its Satake superdiagrams. In the Dynkin diagram of B, we denote the root α_i by the usual white, grey and black dots, and the roots β_l by black dots. We should note that this black dot is different from the black dot associated with a nondegenerate odd root such as in $B(0, n)$ for instance. If $\pi(i) = k$, then it will be indicated by a double-headed arrow $\longleftrightarrow$. We avoid blackening of grey dots, to avoid the uniqueness of the Satake diagram being lost. The involutive automorphisms of these algebras can be determined from their respective Satake superdiagrams. The classification of such involutive automorphisms has been discussed by Levestein [15] and Cornwell [16] for affine Kac–Moody algebras. However, we do not go into these details here, since such a classification is not required for the purposes of our calculation. As an illustration, we restrict ourselves to the consideration of two examples, $A^{(1)}(0, 1)$ and $B^{(1)}(1, 1)$, to highlight the salient features of the scheme.

The Iwasawa decomposition of an affine Kac–Moody algebra $\underset{\sim}{\mathcal{G}}$ is obtained by combining the Cartan decomposition and the root space decomposition of $\underset{\sim}{\mathcal{G}}$. The Cartan decomposition corresponding to $\underset{\sim}{\mathcal{G}}$ is given by

$$\underset{\sim}{\mathcal{G}} = k \oplus p \tag{2.2}$$

Table 1. Satake superdiagrams and involutive automorphisms of $A^{(1)}(0,1)$.

Satake superdiagrams of $A^{(1)}(0,1)$		Involutive root automorphisms
(i)		$-\sigma(\alpha_0)=\alpha_0$ $-\sigma(\alpha_1)=\alpha_1$ $-\sigma(\alpha_2)=\alpha_2$
(ii)		$-\sigma(\alpha_0)=\alpha_1$ $-\sigma(\alpha_1)=\alpha_0$ $-\sigma(\alpha_2)=\alpha_2$
(iii)		$-\sigma(\alpha_0)=\alpha_1+\alpha_2$ $-\sigma(\alpha_1)=\alpha_0+\alpha_2$ $\sigma(\alpha_2)=\alpha_2$

where $\underset{\sim}{k}$ is a maximal compact subalgebra of $\underset{\sim}{\mathcal{G}}$ defined such that $x \in \underset{\sim}{k}$ iff $\sigma x = x$ and $\underset{\sim}{p}$ is a subspace of $\underset{\sim}{\mathcal{G}}$ such that $x \in \underset{\sim}{p}$ iff $\sigma x = -x$. Now applying the root space decomposition of $\underset{\sim}{p}$ we obtain

$$\underset{\sim}{\mathcal{G}} = \underset{\sim}{k} \oplus \underset{\sim}{a} \oplus \underset{\sim}{n} \tag{2.3}$$

where $\underset{\sim}{a}$ is a maximal Abelian subalgebra of $\underset{\sim}{p}$ and $\underset{\sim}{n}$ is a nilpotent subalgebra of $\underset{\sim}{\mathcal{G}}$. The details of the method for direct determination of the Iwasawa decomposition are elaborated elsewhere [3,4,17]. The involutive automorphisms required for this purpose can be determined from the Satake superdiagrams of affine Kac–Moody superalgebras.

3.1. Satake superdiagrams and Iwasawa decomposition of $A^{(1)}(0,1)$

The Cartan matrix of $A^{(1)}(0,1)$ is

$$c = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The three simple roots of $A^{(1)}(0,1)$ are α_0, α_1 and α_2, where $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$. The possible Satake superdiagrams of $A^{(1)}(0,1)$ along with their root automorphisms are represented in table 1.

Let us now consider the involutive automorphism of $A^{(1)}(0,1)$ determined by any one of the Satake superdiagrams, say superdiagram (iii) of table 1. The simple root automorphisms are given by

$$\begin{aligned} -\sigma(\alpha_0) &= \alpha_1 + \alpha_2 \\ -\sigma(\alpha_1) &= \alpha_0 + \alpha_2 \\ \sigma(\alpha_2) &= \alpha_2. \end{aligned} \tag{3.1}$$

In terms of roots α_s' of associated Lie superalgebra $A(0,1)$ and root δ these can be rewritten as

$$\begin{aligned} -\sigma(\alpha_1) &= -\alpha_1 + \delta \\ -\sigma(\delta - \alpha_1 - \alpha_2) &= \alpha_1 + \alpha_2 \\ \sigma(\alpha_2) &= \alpha_2. \end{aligned} \tag{3.2}$$

The positive roots of $A^{(1)}(0,1)$ are given by

$$\Delta = \{\alpha_0, \alpha_1, \alpha_2, \pm\alpha_1 + n\delta, \pm\alpha_2 + n\delta, \pm(\alpha_1 + \alpha_2) + n\delta, n\delta, \text{ where } \quad n \in \mathbb{Z}^+\}. \tag{3.3}$$

We can apply the simple root automorphism to deduce the automorphisms of other roots and we see that

$$\exp\alpha(h) = +1 \qquad \text{for} \quad \alpha = \alpha_0, \alpha_1, \alpha_1 + n\delta, \alpha_2 + n\alpha_0, -(\alpha_1 + \alpha_2) + n\delta \tag{3.4}$$

$$\exp\alpha(h) = -1 \qquad \text{for} \quad \alpha = -\alpha_1 + n\delta, -\delta_2 + n\delta, (\alpha_1 + \alpha_2) + n\delta. \tag{3.5}$$

For $A^{(1)}(0,1)$ $\underset{\sim}{k}$ and $\underset{\sim}{p}$ are given by

$$\underset{\sim}{k} = \{\mathrm{i}h_\alpha \text{ for } \alpha = \alpha_0, \alpha_1, \alpha_2 \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by equation (3.4)}\} \tag{3.6}$$

$$\underset{\sim}{p} = \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by equation (3.5)}\}. \tag{3.7}$$

We now select a maximal Abelian subalgebra $\underset{\sim}{a}$ in the vector space $\underset{\sim}{p}$. It is clear that $\underset{\sim}{a}$ is one-dimensional and may be chosen to have a basis element

$$H_0' = \mathrm{i}(e_\alpha + e_{-\alpha}) \qquad \text{for} \quad \alpha = -\alpha_2 + m\delta, m \in \mathbb{Z}^+. \tag{3.8}$$

So, we have $R_A = \{-\alpha_2 + m\delta\}$ and R_M is empty. $\underset{\sim}{m}$ is two-dimensional and its basis elements are given by

$$\begin{aligned} H_1' &= -(h_{\alpha_2+(m+n)\delta} - 2h_{\alpha_1+(m+n)\delta}) \\ H_2' &= -h_{(m+n)\delta}. \end{aligned} \tag{3.9}$$

Note that H_0', H_1', H_2' together with a scaling element d' such that $\alpha_0(d') = 1$ and $\alpha_i(d') = 0$, for $i = 1, 2, \ldots, l$, are the elements of the Cartan subalgebra H'. The inner automorphism of $A^{(1)}(0,1)$ is given by the expression

$$V = V_{-\alpha_2+n\delta} = \exp[\mathrm{ad}\{\mathrm{i}a_{-\alpha_2+n\delta}(e_{-\alpha_2+n\delta} - e_{\alpha_2+n\delta})\}] \tag{3.10}$$

where

$$a_{-\alpha_2+n\delta} = \frac{\pi}{t^n[8(\alpha_2, \alpha_2)]^{1/2}} \tag{3.11}$$

where t is a complex variable.

Applying this to the Cartan subalgebra H' of $A^{(1)}(0,1)$, we obtain

$$\begin{aligned} H_0 &= 2^{1/2}h_{-\alpha_2+(m+n)\delta} \\ H_1 &= -(h_{\alpha_2+(m+n)\delta} - 2h_{\alpha_1+(m+n)\delta}) \\ H_2 &= -h_{(m+n)\delta} \end{aligned}$$

and

$$d = V_{-\alpha_2+n\delta}(d'). \tag{3.12}$$

With respect to this Cartan subalgebra, the set of positive roots is given by

$$\Delta^+ = \{\alpha_0, \alpha_1, -\alpha_2, \alpha_1 \pm (m+n)\delta, -\alpha_2 \pm (m+n)\delta, -(\alpha_1+\alpha_2) \pm (m+n)\delta, -(m+n)\delta\} \tag{3.13}$$

where Δ_+^+ and Δ_-^+ are given as follows:

$$\begin{aligned} \Delta_-^+ &= \{-(m+n)\delta\} \\ \Delta_+^+ &= \{\alpha_0, \alpha_1, -\alpha_2, \alpha_1 \pm (m+n)\delta, -\alpha_2 \pm (m+n)\delta, -(\alpha_1+\alpha_2) \pm (m+n)\delta\}. \end{aligned} \tag{3.14}$$

The elements of $\tilde{\underset{\sim}{n}}$ are determined by the structure $V^{-1}_{-\alpha_2+n\delta} e_\alpha$ (here $\alpha \in \Delta_+^+$). These are given by

$$\begin{aligned} V^{-1}_{-\alpha_2+n\delta} e_{-\alpha_1 \pm (m+n)\delta} &= \frac{1}{1+t^{2n}} (\operatorname{sgn} N_{-\alpha_2, \alpha_1+\alpha_2}) e_{\alpha_1+\alpha_2+(n\pm(m+n))\delta} + \frac{1}{1+t^{2n}} e_{\alpha_1 \pm (m+n)\delta} \\ V^{-1}_{-\alpha_2+n\delta} e_{-\alpha_2 \pm (m+n)\delta} &= -\frac{\mathrm{i}}{2^{1/2}} h_{\alpha_2 \pm (m+n)\delta} - \frac{1}{2} (e_{\alpha_2+(n\pm(m+n))\delta} - e_{-\alpha_2-(n\pm(m+n))\delta}) \\ V'_{-\alpha_2+n\delta} e_{-(\alpha_1+\alpha_2)\pm(m+n)\delta} &= \frac{1}{1+t^{2n}} (\operatorname{sgn} N_{-\alpha_2, -\alpha_1}) e_{-\alpha_1+(n\pm(m+n))\delta} \\ &\quad + \frac{1}{1+t^{2n}} e_{-(\alpha_1+\alpha_2)\pm(m+n)\delta}. \end{aligned} \tag{3.15}$$

Now $\underset{\sim}{n}$ can be calculated from $\tilde{\underset{\sim}{n}}$, and its elements have the following structures:

$$\begin{aligned} & -\frac{1}{2}(e_{\alpha_2+(n\pm(m+n))\delta} - e_{-\alpha_2+(n\pm(m+n))\delta}) + \frac{\mathrm{i}}{2^{1/2}} h_{\alpha_2+(m+n)\delta} \\ & \frac{1}{1+t^{2n}} (e_{\alpha_1+(n\pm(m+n))\delta} - e_{-\alpha_1+(n\pm(m+n))\delta}) + \frac{1}{1+t^{2n}} (\operatorname{sgn} N_{-\alpha_2, \alpha_1+\alpha_2}) \\ & \qquad \times (e_{\alpha_1+\alpha_2+(n\pm(m+n))\delta} - e_{-\alpha_1-\alpha_2+(n\pm(m+n))\delta}) \\ & -\frac{1}{1+t^{2n}} (\operatorname{sgn} N_{-\alpha_2, -\alpha_1}) (e_{\alpha_1+(n\pm(m+n))\delta} - e_{-\alpha_1+(n\pm(m+n))\delta}) \\ & \qquad + \frac{1}{1+t^{2n}} (e_{\alpha_1+\alpha_2\pm(m+n)\delta} - e_{-(\alpha_1+\alpha_2)\pm(m+n)\delta}). \end{aligned} \tag{3.16}$$

The required Iwasawa decomposition is then written as

$$A^{(1)}(0,1) = \underset{\sim}{k} \oplus \underset{\sim}{a} \oplus \underset{\sim}{n} \tag{3.17}$$

where $\underset{\sim}{k}$, $\underset{\sim}{a}$ and $\underset{\sim}{n}$ are given by equations (3.6), (3.8) and (3.16), respectively.

3.2. Satake superdiagrams and Iwasawa decomposition of $B^{(1)}(1,1)$

The Cartan matrix of $B^{(1)}(1,1)$ is

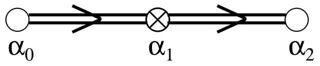
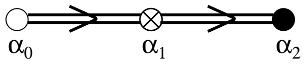
$$c = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

and the three simple roots of $B^{(1)}(1,1)$ are α_0, α_1 and α_2, where $\alpha_0 = \delta - (2\alpha_1 + 2\alpha_2)$. The possible Satake superdiagrams of $B^{(1)}(1,1)$ along with their root automorphisms can be read from table 2.

As a second example, we consider the involutive automorphism of $B^{(1)}(1,1)$ determined by the Satake superdiagrams, say superdiagram (ii) of table 2. The simple root automorphisms are similarly given by

$$\begin{aligned} -\sigma(\alpha_0) &= \alpha_0 \\ -\sigma(\alpha_1) &= \alpha_1 + 2\alpha_2 \\ \sigma(\alpha_2) &= \alpha_2. \end{aligned} \tag{3.18}$$

Table 2. Satake superdiagrams and involutive automorphisms of $B^{(1)}(1, 1)$.

	Satake superdiagrams of $B^{(1)}(1, 1)$	Involutive root automorphisms
(i)		$\sigma(\alpha_0) = \alpha_0$ $\sigma(\alpha_1) = \alpha_1$ $-\sigma(\alpha_2) = \alpha_2$
(ii)		$-\sigma(\alpha_0) = \alpha_0$ $-\sigma(\alpha_1) = \alpha_1 + 2\alpha_2$ $\sigma(\alpha_2) = \alpha_2$

These automorphisms can be written as

$$\begin{aligned} &-\sigma(\alpha_1) = -\alpha_1 + 2\alpha_2 \\ &-\sigma(\delta - 2\alpha_1 - 2\alpha_2) = \delta - 2\alpha_1 - 2\alpha_2 \\ &\sigma(\alpha_2) = \alpha_2. \end{aligned} \tag{3.19}$$

The positive roots of $B^{(1)}(1, 1)$ are

$$\begin{aligned} \Delta = \{&\alpha_0, \alpha_1, \alpha_2, \pm\alpha_1 + n\delta, \pm\alpha_2 + n\delta, \pm(\alpha_1 + \alpha_2) + n\delta, \\ &\pm(\alpha_1 + 2\alpha_2) + n\delta, \pm(2\alpha_1 + 2\alpha_2) + n\delta, n\delta, \text{ where } n \in \mathbb{Z}^+\}. \end{aligned} \tag{3.20}$$

We can apply the simple root automorphisms to find the automorphisms of other roots such that

$$\exp\alpha(h) = +1 \qquad \text{for} \quad \alpha = \alpha_2, \alpha_2 + n\delta, \pm(\alpha_1 + 2\alpha_2) + n\delta \tag{3.21}$$

$$\begin{aligned} \exp\alpha(h) = -1 \qquad \text{for} \quad &\alpha = \alpha_0, \alpha_1, \pm\alpha_1 + n\delta, -\alpha_2 + n\delta, \pm(\alpha_1 + \alpha_2) + n\delta, \\ &\pm(2\alpha_1 + 2\alpha_2) + n\delta, n\delta. \end{aligned} \tag{3.22}$$

For $B^{(1)}(1, 1)$ $\underset{\sim}{k}$ and $\underset{\sim}{p}$ are given by

$$\underset{\sim}{k} = \{\mathrm{i}h_\alpha \text{ for } \alpha = \alpha_0, \alpha_1, \alpha_2 \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by equation (3.21)}\} \tag{3.23}$$

$$\underset{\sim}{p} = \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by equation (3.22)}\}. \tag{3.24}$$

Now $\underset{\sim}{a}$ is one-dimensional and its basis element is chosen to be

$$H_0' = \mathrm{i}(e_\alpha + e_{-\alpha}) \qquad \text{for} \quad \alpha = \alpha_1 + \alpha_2. \tag{3.25}$$

Here $R_A = \{\alpha_1 + \alpha_2\}$ and R_M is again empty. $\underset{\sim}{m}$ is two-dimensional and its basis elements have the form

$$\begin{aligned} H_1' &= h_{\alpha_2} \\ H_2' &= -h_\delta. \end{aligned} \tag{3.26}$$

The elements H_0', H_1' and H_2' together with a scaling element d' form the Cartan subalgebra $\underset{\sim}{h}'$. Defining the inner automorphism as $V_{\alpha_1+\alpha_2}$ and applying this to the Cartan subalgebra $\underset{\sim}{h}'$, we obtain

$$\begin{aligned} H_0 &= -2^{1/2} h_{\alpha_1+\alpha_2} \\ H_1 &= h_{\alpha_2} \\ H_2 &= -h_\delta \\ d &= V_{\alpha_2+\alpha_2}(d'). \end{aligned} \tag{3.27}$$

With respect to this Cartan subalgebra, the positive roots of $B^{(1)}(1, 1)$ are given by

$$\Delta = \{-\alpha_0, \alpha_1, -\alpha_2, \alpha_1 \pm n\delta, -\alpha_2 \pm n\delta, (\alpha_1 + \alpha_2) \pm n\delta, \\ (\alpha_1 + 2\alpha_2) \pm n\delta, (2\alpha_1 + 2\alpha_2) \pm n\delta, -n\delta\}. \tag{3.28}$$

The Δ_-^+ and Δ_+^+ are then written as

$$\Delta_-^+ = \{-\alpha_2, -\alpha_2 \pm n\delta, -n\delta\} \tag{3.29}$$

$$\Delta_+^+ = \{-\alpha_0, \alpha_1, \alpha_1 \pm n\delta, (\alpha_1 + \alpha_2) \pm n\delta, (\alpha_1 + 2\alpha_2) \pm n\delta, (2\alpha_1 + 2\alpha_2) \pm n\delta\}. \tag{3.30}$$

The elements of $\tilde{\underset{\sim}{n}}$ are determined by the structures $V^{-1}_{\alpha_1+\alpha_2} e_\alpha$, where $\alpha \in \Delta_+^+$. These are given by

$$\begin{aligned}
V^{-1}_{\alpha_1+\alpha_2} e_{\alpha_1 \pm n\delta} &= \frac{1}{2} e_{\alpha_1} - \frac{\mathrm{i}}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,-\alpha_2}) e_{-\alpha_2} \\
V^{-1}_{\alpha_1+\alpha_2} e_{\alpha_1 \pm n\delta} &= \tfrac{1}{2} e_{\alpha_1 \pm n\delta} - \tfrac{1}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,-\alpha_2}) e_{-\alpha_2 \pm n\delta} \\
V^{-1}_{\alpha_1+\alpha_2} e_{(\alpha_1+\alpha_2)\pm n\delta} &= \frac{1}{2}(e_{(\alpha_1+\alpha_2)\pm n\delta} - e_{-(\alpha_1+\alpha_2)\pm n\delta}) - \frac{\mathrm{i}}{2^{1/2}} h_{(\alpha_1+\alpha_2)\pm n\delta} \\
V^{-1}_{\alpha_1+\alpha_2} e_{(\alpha_1+2\alpha_2)\pm n\delta} &= \frac{-\mathrm{i}}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,\alpha_2}) e_{\alpha_2 \pm n\delta} + \frac{1}{2} e_{(\alpha_1+2\alpha_2)} \pm n\delta \\
V^{-1}_{\alpha_1+\alpha_2} e_{(2\alpha_1+2\alpha_2)\pm n\delta} &= -\frac{\mathrm{i}}{2}\mathrm{sgn}\ (N_{\alpha_1+\alpha_2,\alpha_1+\alpha_2}) e_{(\alpha_1+\alpha_2)\pm n\delta} + \frac{1}{2} e_{(2\alpha_1+2\alpha_2)\pm n\delta}.
\end{aligned} \tag{3.31}$$

The elements of $\underset{\sim}{n}$ can be obtained from $\tilde{\underset{\sim}{n}}$ which are given by the following expressions:

$$\begin{aligned}
&\frac{1}{2}(e_{(\alpha_1+\alpha_2)\pm n\delta} - e_{-(\alpha_1+\alpha_2)\pm n\delta}) - \frac{\mathrm{i}}{2^{1/2}} h_{(\alpha_1+\alpha_2)\pm n\delta} \\
&\frac{1}{2}(e_{\alpha_1} - e_{-\alpha_1}) + \frac{\mathrm{i}}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,\alpha_2})(e_{\alpha_2} - e_{-\alpha_2}) \\
&\frac{1}{2}(e_{\alpha_1\pm n\delta} - e_{-\alpha_1\pm n\delta}) + \frac{\mathrm{i}}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,-\alpha_2})(e_{\alpha_2\pm n\delta} - e_{-\alpha_2\pm n\delta}) \\
&\frac{1}{2}(e_{(\alpha_1+2\alpha_2)\pm n\delta} - e_{-(\alpha_1+2\alpha_2)\pm n\delta} - \frac{\mathrm{i}}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,\alpha_2})(e_{\alpha_2\pm n\delta} - e_{-\alpha_2\pm n\delta}) \\
&\frac{1}{2}(e_{(2\alpha_1+2\alpha_2)\pm n\delta} - e_{-(2\alpha_1+2\alpha_2)\pm n\delta}) - \frac{\mathrm{i}}{2}(\mathrm{sgn}\ N_{\alpha_1+\alpha_2,\alpha_1+\alpha_2})(e_{(\alpha_1+\alpha_2)\pm n\delta} - e_{-(\alpha_1+\alpha_2)\pm n\delta}).
\end{aligned} \tag{3.32}$$

The required Iwasawa decomposition is now

$$B^{(1)}(1, 1) = \underset{\sim}{k} \oplus \underset{\sim}{a} \oplus \underset{\sim}{n} \tag{3.33}$$

where $\underset{\sim}{k}$, $\underset{\sim}{a}$ and $\underset{\sim}{n}$ are given by relations (3.23), (3.25) and (3.32), respectively.

4. Discussion

A critical examination of the conclusions arrived at above seems to be in order. It has been demonstrated how the technique of Satake diagrams has proved useful in analysing the various important aspects of the algebraic structures, including their supersymmetric extensions. The Iwasawa decomposition of the affine Kac–Moody superalgebras immediately leads to the Langlands decomposition of these algebras, thereby facilitating the determination of the parabolic subalgebras that are necessary for obtaining the corresponding induced representations with the help of Schmidt construction. However, we note in passing that in general there are $2^{|m_1|}$ classes of parabolic subalgebras, with $|m_1|$ as the dimension of $\underset{\sim}{a}$. For the two illustrative examples $A^{(1)}(0, 1)$ and $B^{(1)}(1, 1)$ considered here, we have $|m_1| = 1$ and there will, therefore, be at most two parabolic subalgebras, one being the minimal parabolic and the other the algebra itself. For the sake of generality, it would be quite interesting to determine the parabolic subalgebras in cases where $|m_1| > 1$. Such studies for the affine Kac–Moody superalgebras are currently in progress and will be reported in a subsequent communication.

Acknowledgment

It is a pleasure to thank Dr R S Kaushal for enlightening discussions and helpful comments on the manuscript.

Appendix

The general formulation of inner automorphism appropriate for the case of Kac–Moody algebras along with their supersymmetric extensions is defined as

$$V = V_{\pm\alpha+n\delta} = \exp\{\mathrm{ad}[\mathrm{i}a_{\pm\alpha+n\delta}(e_{\pm\alpha+n\delta} - e_{\mp\alpha+n\delta})]\} \tag{A.1}$$

where

$$a_{\pm\alpha+n\delta} = \frac{\pi}{t^n(8(\alpha,\alpha))^{1/2}}. \tag{A.2}$$

Using the formula

$$[\exp(\mathrm{ad}\ A)]B = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \cdots$$

and equation (A.1) we see that

$$\begin{aligned} V_{\pm\alpha+n\delta}[\mathrm{i}(e_{\alpha+m\delta} - e_{-\alpha+m\delta})] &= \mathrm{i}(e_{\alpha+m\delta} - e_{-\alpha+m\delta}) \\ V_{\pm\alpha+n\delta}(e_{\alpha+m\delta} - e_{-\alpha+m\delta}) &= \pm\mathrm{i}\left(\frac{2}{(\alpha,\alpha)}\right)^{1/2} h_{\alpha+(m+n)\delta} \\ V_{\pm\alpha+n\delta}(\mathrm{i}h_{\alpha+m\delta}) &= -\left(\frac{(\alpha,\alpha)}{2}\right)^{1/2}(e_{\alpha+(m+n)\delta} - e_{-\alpha+(m+n)\delta}). \end{aligned} \tag{A.3}$$

From these expressions we find

$$V^{-1}_{\pm\alpha+n\delta}e_{\alpha\pm(m+n)\delta} = \frac{1}{2}(e_{\alpha\pm(m+n)\delta} - e_{-\alpha\pm(m+n)\delta}) \mp \frac{\mathrm{i}}{2}\left(\frac{2}{(\alpha,\alpha)}\right)^{1/2} h_{\alpha\pm m\delta} \tag{A.4}$$

$$V^{-1}_{\pm\alpha+n\delta}e_{-\alpha\pm(m+n)\delta} = \mp\frac{\mathrm{i}}{2}\left(\frac{2}{(\alpha,\alpha)}\right)^{1/2} h_{\alpha\pm m\delta} - \frac{1}{2}(e_{\alpha\pm(m+n)\delta} - e_{-\alpha\pm(m+n)\delta}). \tag{A.5}$$

For any γ which is not an 'α' string root, we have

$$V^{-1}_{\pm\alpha+n\delta}e_{\gamma\pm m\delta} = e_{\gamma\pm m\delta}. \tag{A.6}$$

We now consider the case when the 'α' string containing γ has two members and γ as the first member of the string. Then we have

$$V^{-1}_{\pm\alpha+n\delta}e_{\gamma\pm m\delta} = \left(\frac{1}{1+t^{2n}}\right)e_{\gamma\pm n\delta} \mp \left(\frac{1}{1+t^{2n}}\right)(\mathrm{sgn}\ N_{\alpha,\gamma})e_{\gamma+\alpha\pm(m+n)\delta} \tag{A.7}$$

and when γ is the second member of the string, the resulting expression is

$$V^{-1}_{\pm\alpha+n\delta}e_{\gamma\pm m\delta} = \mp\left(\frac{1}{1+t^{2n}}\right)(\mathrm{sgn}\ N_{\alpha,\gamma-\alpha})e_{\gamma-\alpha\pm(m+n)\delta} + \left(\frac{1}{1+t^{2n}}\right)e_{\gamma\pm m\delta}. \tag{A.8}$$

In arriving at these relations we have been guided by the work of Cornwell [17], coupled with the fact that the structure constants of affine Kac–Moody algebras depend only on the structure constants of the associated Lie algebras. That is to say,

$$N_{\alpha+m\delta,\beta+n\delta} = N_{\alpha,\beta}. \tag{A.9}$$

In a similar fashion, we can derive the structure $V_\alpha^{-1}e_\gamma$ for all other cases.

References

[1] Kac V G 1985 *Infinite Dimensional Lie Algebras* 2nd edn (Cambridge: Cambridge University Press)
[2] Goddard P and Olive D 1986 *Int. J. Mod. Phys.* A **1** 303
[3] Pati K C and Parashar D 1998 *J. Math. Phys.* **39** 5015
Pati K C and Parashar D 1998 *J. Phys. A: Math. Gen.* **31** 767
[4] Pati K C, Parashar D and Kaushal R S 1999 *J. Math. Phys.* **40** 501
[5] Loos O 1969 *Symmetric Spaces* vol 2 (New York: Benjamin)
[6] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
[7] McDonald I G 1977 *Proc. SRC/LMS Research Symp. on Representations of Lie Groups* ed M F Atiyah (Oxford: Clarendon) p 91
[8] Kac V G 1977 *Adv. Math.* **26** 8
[9] Frappat L, Sciarrino A and Sorba P 1989 *Commun. Math. Phys.* **121** 459
[10] Scheunert M 1979 *Lecture Notes in Mathematics* vol 716 (Berlin: Springer)
[11] Parker M 1980 *J. Math. Phys.* **21** 689
[12] Serganova V V 1983 *Funct. Anal. Appl.* **17** 46 (Engl. transl. p 200)
[13] Serganova V V 1985 *Math. USSR Izv.* **24** 539
[14] Ziranbauer M R 1996 *J. Math. Phys.* **37** 4986
[15] Levestein F 1988 *J. Algebra* **114** 489
[16] Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 2311
Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 2335
Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 3955
[17] Cornwell J F 1975 *J. Math. Phys.* **16** 1992
Cornwell J F 1979 *J. Math. Phys.* **20** 547